CHAPTER 17
PLANE FIGURES

The discussion of lines and planes in chapter 1 of this course was limited to their consideration as examples of sets. The present chapter is concerned with lines, angles, and areas as found in various plane (flat) geometric figures.

LINES

In the strictly mathematical sense, the term "line segment" should be used whenever we refer to the straight line joining some point A to some other point B. However, since the straight lines comprising geometric figures have clearly designated end points, we may simplify our terminology. Throughout the remaining chapters of this course, the general term "line" is used to designate straight line segments, unless stated otherwise.

TYPES OF LINES

The two basic types of lines in geometry are straight lines and curved lines. A curved line joining points A and B is designated as "curve AB." (See fig. 17-1.) If curve AB is an arc of a circle, it may be designated as "arc AB."

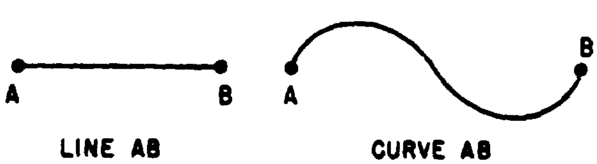

Figure 17-1.—Straight and curved lines.

The term "broken line" in mathematics means a series of two or more straight segments connected end-to-end but not running in the same direction. In mathematics, a series of short, straight segments with breaks between them, which would form a single straight line if joined end-to-end, is a DASHED LINE. (See fig. 17-2.)

Figure 17-2.—Broken and dashed lines.

ORIENTATION

Straight lines may be classified in terms of their orientation to the observer's horizon or in terms of their orientation to each other. For example, lines in the same plane which run beside each other without meeting at any point, no matter how far they are extended, are PARALLEL. (See fig. 17-3 (A).) Lines in the same plane which are not parallel are OBLIQUE. Oblique lines meet to form angles (discussed in the following section). If two oblique lines cross or meet in such a way as to form four equal angles, as in figure 17-3 (B), the lines are PERPENDICULAR. This definition includes the case in which only one angle is formed, such as angle AEC in figure 17-3 (C). By extending line AE to form line AD, and extending CE to form CB, four equal angles (AEC, CED, DEB, and BEA) are formed.

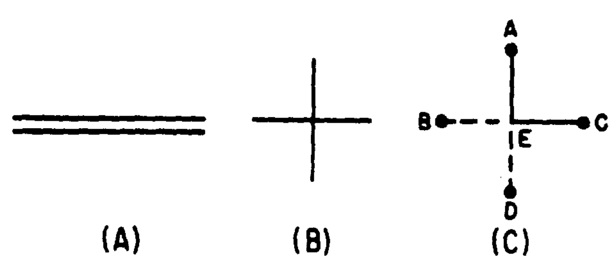

Figure 17-3.—(A) Parallel lines; (B) and (C) perpendicular lines.

Lines parallel to the horizon are HORIZONTAL. Lines perpendicular to the horizon are VERTICAL.

ANGLES

Lines which meet or cross each other are said to INTERSECT. Angles are formed when two straight lines intersect. The two lines which form an angle are its SIDES, and the point where the sides intersect is the VERTEX. In figure 17-4, the sides of the angles are AV and BV, and the vertex is V in each case. Figure 17-4 (A) is an ACUTE angle; (B) is an OBTUSE angle.

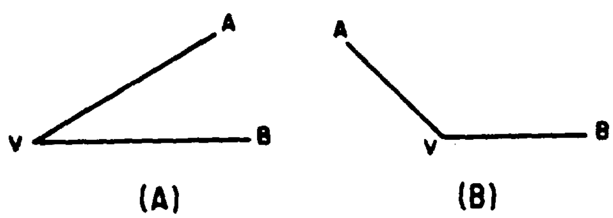

Figure 17-4.—(A) Acute angle; (B) obtuse angle.

CLASSIFICATION BY SIZE

When the sides of an angle are perpendicular to each other, the angle is a RIGHT angle. This term is related to the Latin word "rectus," which may be translated "erect" or "upright." Thus, if one side of a right angle is horizontal, the other side is erect or upright.

The size of an angle refers to the amount of separation between its sides, and the unit of angular size is the angular DEGREE. A right angle contains 90 degrees, abbreviated 90°. An angle smaller than a right angle is acute; an angle larger than a right angle is obtuse. Therefore, acute angles are angles of less than 90°, and obtuse angles are angles between 90° and 180°.

If side AV in figure 17-5 (A) is moved downward, the size of the obtuse angle AVB is increased. If side AV is moved so far that it coincides with (lies on top of) CV as in figure 17-5 (B), an angle is formed which is equal to the sum of two right angles. The special angle thus formed (AVB) is a straight angle, so called because it is visually indistinguishable from a straight line.

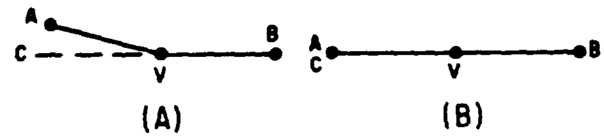

Figure 17-5.—(A) Large obtuse angle; (B) straight angle.

GEOMETRIC RELATIONSHIPS

Angles are often classified by their relationship to other angles or to other parts of a geometric figure. For example, angles 1 and 3 in figure 17-6 are VERTICAL angles, so called because they share a common vertex. Angles 2 and 4 are opposite each other and are also vertical angles. Lines which cross, as in figure 17-6, always form two pairs of vertical angles, and the vertical angles thus formed are equal in pairs; that is, angle 1 equals angle 3, and angle 2 equals angle 4.

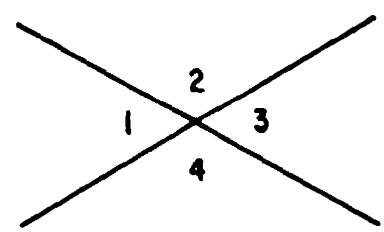

Figure 17-6.—Vertical angles.

Angles 1 and 2 in figure 17-6 are ADJACENT angles. Other pairs of adjacent angles in figure 17-6 are 2 and 3, 3 and 4, and 1 and 4. In the sense used here, adjacent means side by side, not merely close together or touching. For example, angles 1 and 3 are not adjacent angles even though they touch each other.

COMPLEMENTS AND SUPPLEMENTS

Two angles whose sum is 90° are complementary. For example, a 60° angle is the complement of a 30° angle, and conversely. "Conversely" is a mathematical word meaning "vice versa." Two angles whose sum is 180° are supplementary. For example, a 100° angle is the supplement of an 80° angle, and conversely.

Practice problems.

1. Describe the angle which is the complement of an acute angle.

2. Describe the angle which is the supplement of a right angle.

3. If two equal angles are complementary, each contains how many degrees?

4. Find the size of an angle which is twice as large as its own complement.

(Hint: If x is the angle, then $90^\circ - x$ is its complement.)

Answers:

1. Acute
2. Right
3. 45°
4. 60°

GEOMETRIC FIGURES

The discussion of geometric figures in this chapter is limited to polygons and circles. A POLYGON is a plane closed figure, the sides of which are all straight lines. Among the polygons discussed are triangles, parallelograms, and trapezoids.

TRIANGLES

A triangle is a polygon which has three sides and three angles. In general, any polygon has as many angles as it has sides, and conversely.

Parts of a Triangle

Each of the three angles of a triangle is a VERTEX; therefore, every triangle has three vertices. The three straight lines joining the vertices are the SIDES (sometimes called legs), and the side upon which the triangle rests is its BASE, often designated by the letter b. This definition assumes that the standard position of a triangle drawn for general discussion is as shown in figure 17-7, in which the triangle is lying on one of its sides. The vertex opposite the base is the highest point of a triangle in standard position, and is thus called the APEX.

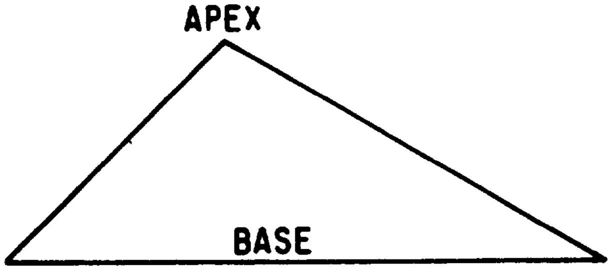

Figure 17-7.—Triangle in standard position.

A straight line perpendicular to the base of a triangle, joining the base to the apex, is the ALTITUDE, often designated by the letter a. The altitude is sometimes referred to as the height, and is then designated by the letter h. Figure 17-8 (B) shows that the apex may not be situated directly above the base. In this case, the base must be extended, as shown by the dashed line, in order to drop a perpendicular from the apex to the base. Mathematicians often use the term "drop a perpendicular." The meaning is the same as "draw a straight, perpendicular line."

In general, the geometrical term "distance from a point to a line" means the length of a perpendicular dropped from the point to the line. Many straight lines could be drawn from a line to a point not on the line, but the shortest of these is the one we use in measuring the

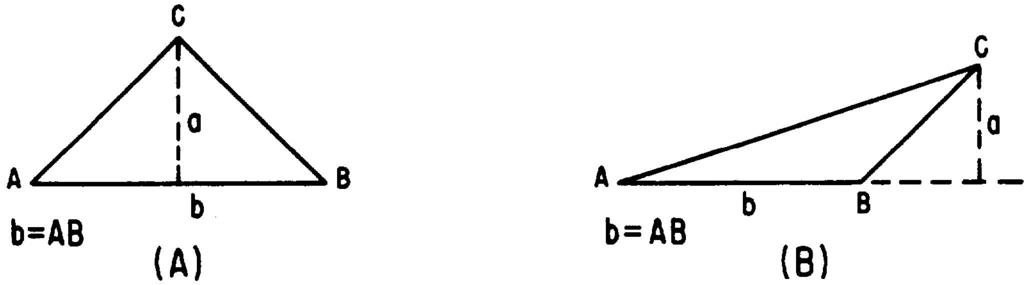

Figure 17-8.—(A) Interior altitude line; (B) exterior altitude line.

distance from the point to the line. The shortest one is perpendicular to the line.

Perimeter and Area

The PERIMETER of a triangle is the sum of the lengths of its sides. In less precise terms, this is sometimes stated as "the distance around the triangle." If the three sides are labeled a, b, and c, the perimeter P can be found by the following formula:

$$P = a + b + c$$

The area of a triangle is the space bounded (enclosed) by its sides. The formula for the area can be found by using a triangle which is part of a rectangle. In figure 17-9, triangle ABC is one-half of the rectangle. Since the area of the rectangle is a times b (that is, ab), the area of the triangle is given by the following formula:

$$\text{Area} = \frac{1}{2}\,ab$$

Written in terms of h, representing height, the formula is:

$$A = \frac{1}{2}\,bh$$

This formula is valid for every triangle, including those with no two sides perpendicular.

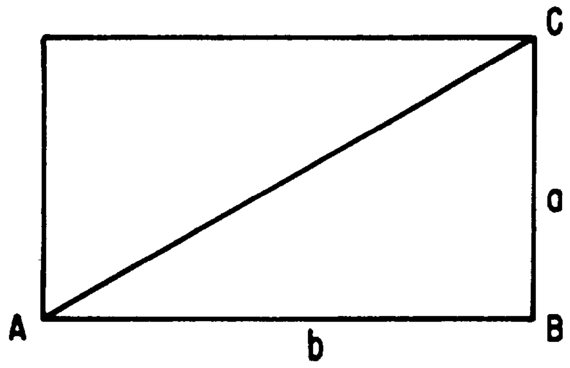

Figure 17-9.—Area of a triangle.

Practice problems. Find the perimeter and area of each of the triangles in figure 17-10.

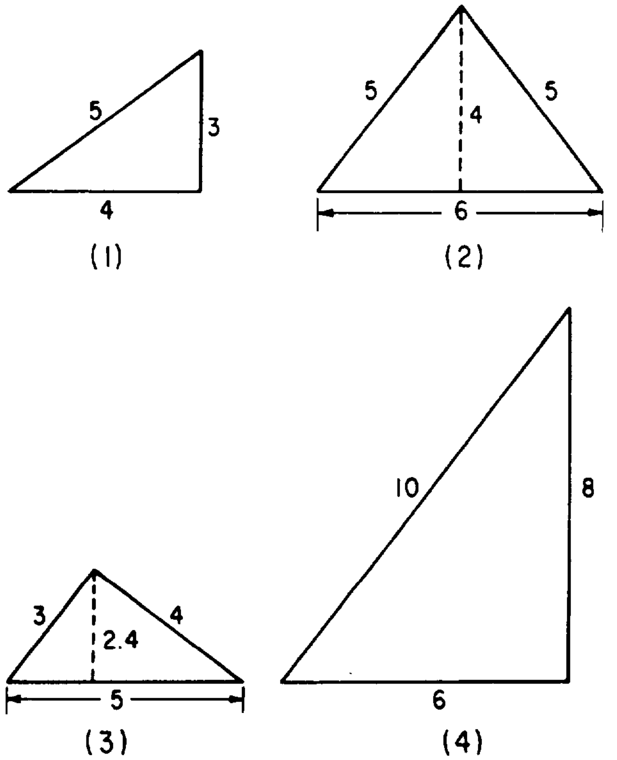

Figure 17-10.—Perimeters and areas of triangles.

Answers:

1. P = 12 units
   A = 6 square units
2. P = 16 units
   A = 12 square units
3. P = 12 units
   A = 6 square units
4. P = 24 units
   A = 24 square units

CAUTION: The concept of area is meaningless if the units of the multiplied dimensions are not the same. For example, if the base of a triangle is 2 feet long and the altitude is 6 inches long, the area might be carelessly stated as $\frac{1}{2}$ (6) (2). However, the units must be considered in order to decide whether the answer is in square feet or square inches. When the units are considered, we realize that the correct answer is

$$\frac{1}{2}\,(6\text{ in.})\,(24\text{ in.}) = 72\text{ sq in.}$$

$$\frac{1}{2}\left(\frac{1}{2}\text{ ft}\right)(2\text{ ft}) = \frac{1}{2}\text{ sq ft}$$

Special Triangles

The classification of triangles depends upon their special characteristics, if any. For example, a triangle may have all three of its sides equal in length; it may have two equal sides and a third side which is longer or shorter than the other two; it may contain a right angle or an obtuse angle. If it has none of these special characteristics, it is a SCALENE triangle. A scalene triangle has no two of its sides equal and no two of its angles equal.

RIGHT TRIANGLE.—If one of the angles of a triangle is a right angle, the figure is a right triangle. The sides which form the right angle are the LEGS of the triangle, and the third side (opposite the right angle) is the HYPOTENUSE.

The area of a right triangle is always easy to determine. If the base of the triangle is one of its legs, as in figure 17-10 (4), the other leg is the altitude. If the hypotenuse is acting as the base, as in figure 17-10 (3), the triangle can be turned until one of its legs is the base, as in figure 17-10 (1). If the triangle is not known to be a right triangle, then the altitude must be given, as in figure 17-10 (2), in order to calculate the area.

Any triangle whose sides are in the ratio of 3:4:5 is a right triangle. Thus, triangles with sides as follows are right triangles:

Side 1	Side 2	Side 3
3	4	5
6	8	10
12	16	20
3x	4x	5x

(x is any positive number)

In addition to the 3-4-5 triangle, two other types of right triangles occur frequently. Any triangle having one 30° angle and one 60° angle is a right triangle; that is, its third angle is 90°. Any triangle having two 45° angles is a right triangle.

ISOSCELES TRIANGLE.—A triangle having two of its sides equal in length is an ISOSCELES triangle. Since the length of the side opposite an angle is determined by the size of the angle, the isosceles triangle has two equal angles. In figure 17-11 (A), triangle ABC is isosceles. Sides AC and BC are equal in length, and angles A and B are equal.

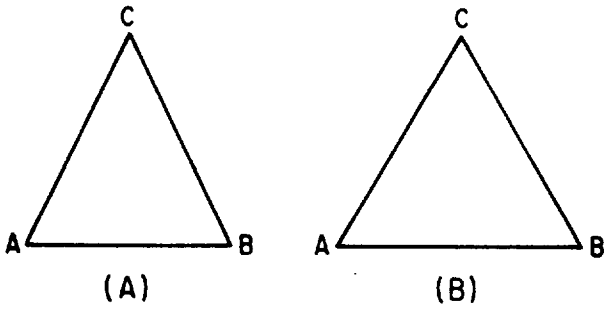

Figure 17-11.—(A) Isoceles triangle; (B) equilateral triangle.

Figure 17-11 (B) illustrates an EQUILATERAL triangle, which is a special case of an isosceles triangle. An equilateral triangle has all three of its sides equal in length. Since the lengths of the sides are directly related to the size of the angles opposite them, an equilateral triangle is also equiangular; that is, all three of its angles are equal.

OBLIQUE TRIANGLES.—Any triangle containing no right angle is an OBLIQUE triangle. Figure 17-12 illustrates two possible configurations, both of which are oblique triangles. An oblique triangle which contains an obtuse angle is often called an OBTUSE triangle.

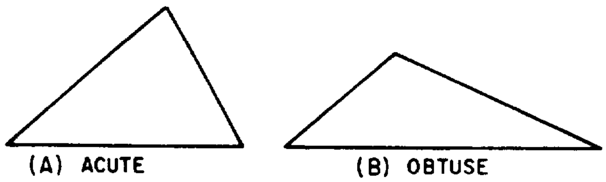

Figure 17-12.—Oblique triangles. (A) Acute; (B) obtuse.

Sum of the Angles

The sum of the angles in any triangle is 180°. For example, if one of the angles is 40° and another is 20°, the third angle is 120°. It is this relationship that justifies the statements made in the preceding section concerning 45° triangles and 30°-60°-90° triangles. If two of the angles are 45° each, then the third angle is 180° - (45°+45°) and the figure is a right triangle. If one angle is 60° and another is 30°, the third angle is 90° and the figure is a right triangle.

QUADRILATERALS

A QUADRILATERAL is a polygon with four sides. The parts of a quadrilateral are its sides, its four angles, and its two DIAGONALS. A diagonal is a straight line joining two alternate vertices of a polygon. Figure 17-13 illustrates the parts of a quadrilateral, in which AC and DB are the diagonals.

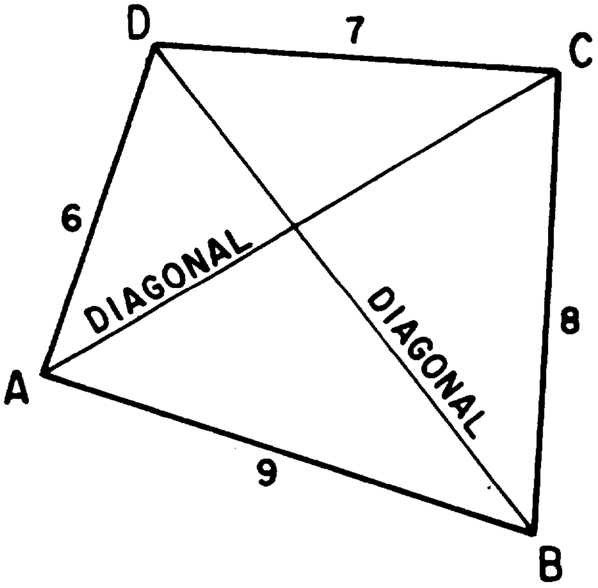

Figure 17-13.—Parts of a quadrilateral.

Perimeter and Area

The perimeter of a quadrilateral is the sum of the lengths of its sides. For example, the perimeter of the quadrilateral in figure 17-13 is 30 units.

The area of a quadrilateral can be found by dividing it into triangles and summing the areas of the triangles. However, the altitudes of the triangles are usually difficult to calculate unless the quadrilateral has at least one pair of parallel sides.

Parallelograms

A PARALLELOGRAM is a quadrilateral in which the opposite sides are parallel. For example, in the parallelogram in figure 17-14, side AB is parallel to side CD. Furthermore, side BC is parallel to side AD.

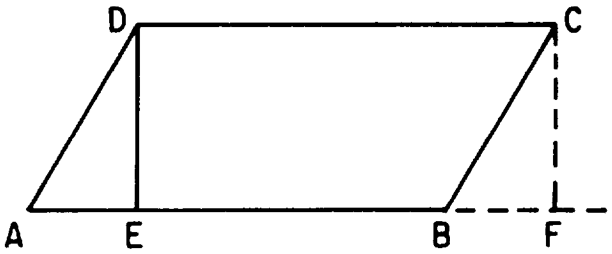

Figure 17-14.—A parallelogram.

Since lines AB and CD are parallel, lines DE and CF (both perpendicular to line AF in figure 17-14) are equal. Angles DAE and CBF in figure 17-14 are equal, because a straight line cutting two parallel lines, such as AD and BC, forms equal angles with the parallel lines. Thus, triangles AED and BFC are equal, and line AD equals line BC. Therefore we have proved that the opposite sides of a parallelogram are equal. If all four of the sides are the same length, the parallelogram is a RHOMBUS.

In addition to the equality of the opposite sides, the opposite angles of a parallelogram are also equal. For example, angle DAB equals angle BCD in figure 17-14, and angle ADC equals angle ABC.

RECTANGLES AND SQUARES.—When all of the angles of a parallelogram are right angles, it is a RECTANGLE. A rectangle with all four of its sides the same length is a SQUARE. Thus a square is a rhombus having 90° angles. Every square is a rectangle, and every rectangle is a parallelogram. Notice that the reverse of this statement is not true.

The area of a rectangle is found by multiplying its length times its width. Therefore, if each side of a square has length s, the area of the square is s^2.

Written as formulas, these areas are as follows:

Rectangle: $A = lw$

or $A = bh$, where b = base, h = height

Square: $A = s^2$

AREA.—The area of a parallelogram can be found by dividing it into rectangles and triangles. For example, in figure 17-14 the area of the parallelogram is the sum of the areas of triangle AED and figure EBCD. Since triangle AED is equal to triangle BFC, the sum of AED and EBCD is equal to the sum of BFC and EBCD. Thus the area of parallelogram ABCD is the same as the area of rectangle EFCD. Since the area of EFCD is DC multiplied by DE, and DC has the same length as AB, we conclude that the area of a parallelogram is the product of its base times its altitude. Written as a formula, this is

$$A = ba$$

or

$$A = bh, \text{ where } h \text{ is height}$$

Trapezoids

A TRAPEZOID is a quadrilateral in which two sides are parallel and the other two sides are not parallel. By orienting a trapezoid so that its parallel sides are horizontal, we may call the parallel sides bases. Observe that the bases of a trapezoid are not equal in length. (See fig. 17-15.)

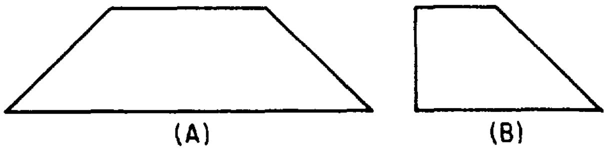

Figure 17-15.—Typical trapezoids.

The area of a trapezoid may be found by separating it into two triangles and a rectangle, as in figure 17-16. The total area A of the trapezoid is the sum of A_1 plus A_2 plus A_3, and is calculated as follows:

$$\begin{aligned} A &= A_1 + A_2 + A_3 \\ &= \frac{1}{2}ha + hb_1 + \frac{1}{2}hc \\ &= \frac{1}{2}h\ (a + 2b_1 + c) \\ &= \frac{1}{2}h\ [(a + b_1 + c) + b_1] \\ &= \frac{1}{2}h\ (b + b_1) \end{aligned}$$

Thus the area of a trapezoid is equal to one-half the altitude times the sum of the bases.

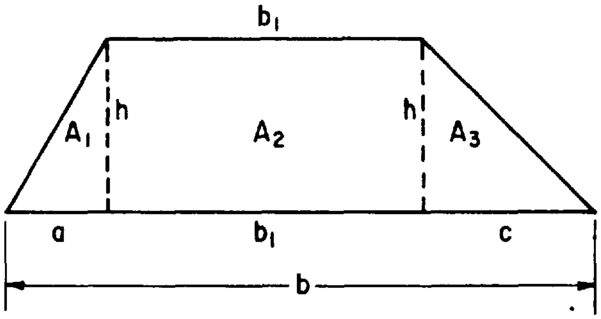

Figure 17-16.—Area of a trapezoid.

Practice problems. Find the area of each of the following figures:

1. Rhombus; base 4 in., altitude 3 in.
2. Rectangle; base 6 ft, altitude 4 ft
3. Parallelogram; base 10 yd, altitude 12 ft
4. Trapezoid; bases 6 ft and 4 ft, altitude 2 yd.

Answers:

1. 12 sq in.	3. 40 sq yd
2. 24 sq ft	4. 30 sq ft

CIRCLES

The mathematical definition of a circle states that it is a plane figure bounded by a curved line, every point of which is equally distant from the center of the figure. The parts of a circle are its circumference, its radius, and its diameter.

Parts of a Circle

The CIRCUMFERENCE of a circle is the line that forms its outer boundary. Circumference is the special term used in referring to the "perimeter" of a circle. (See fig. 17-17.) A RADIUS of a circle is a line joining the center to a point on the circumference, as shown in figure 17-17. A straight line joining two points on the circumference of a circle, and passing through the center, is a DIAMETER. A straight line which touches the circle at just one point is a TANGENT. A tangent is perpendicular to a radius at the point of tangency.

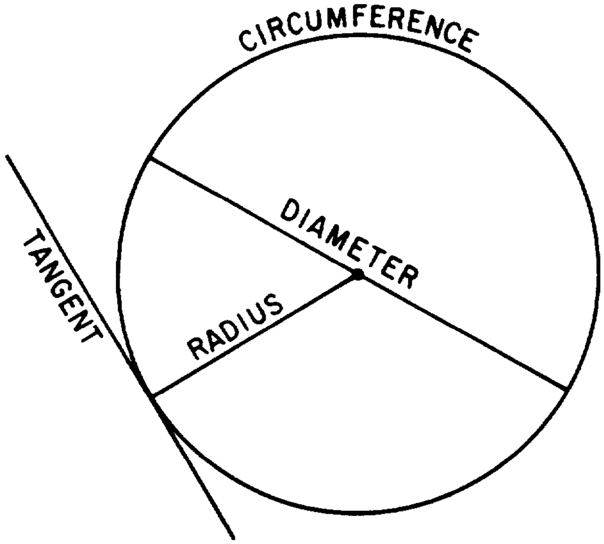

Figure 17-17.—Parts of a circle.

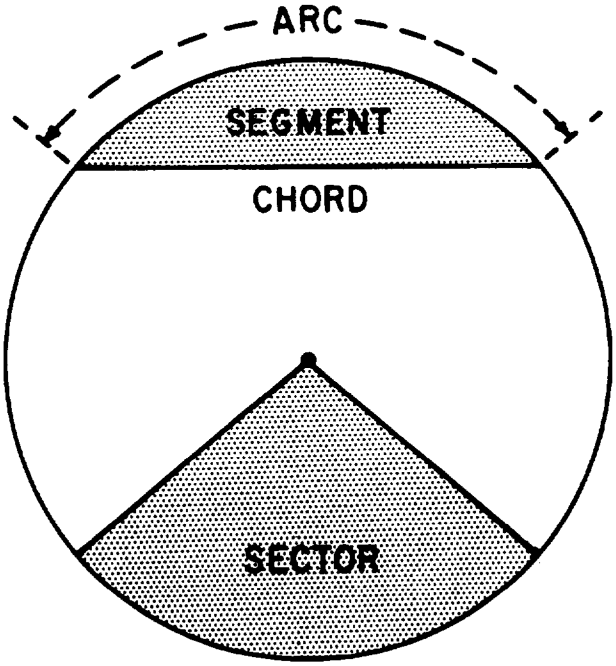

Figure 17-18.—Arc, chord, segment, and sector.

An ARC is a portion of the circumference of a circle. A CHORD is a straight line joining the end points of any arc. The portion of the area of a circle cut off by a chord is a SEGMENT of the circle, and the portion of the circle's area cut off by two radii (radius lines) is a SECTOR. (See fig. 17-18.)

Formulas for Circumference and Area

The formula for the circumference of a circle is based on the relationship between the circumference and the diameter. This comparison can be made experimentally by marking the edge of a circular object, such as a coin, and rolling it (without slippage) along a flat surface. (See fig. 17-19.)

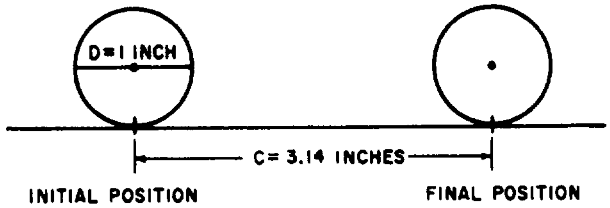

Figure 17-19.—Measuring the circumference of a circle.

The distance from the initial position to the final position of the disk in figure 17-19 is approximately 3.14 times as long as the diameter of the disk. With any circle, this is always found to be the case; but it is not possible to give the value of C/d (circumference divided by diameter) exactly. The ratio C/d is represented by the symbol π, which is the Greek letter pi. Thus we have the following equations:

$$\frac{C}{d} = \pi$$

$$C = \pi d$$

This formula states that the circumference of a circle is π times the diameter. Notice that it could be written as

$$C = 2r \cdot \pi \text{ or } C = 2\pi r$$

since the diameter d is the same as 2r (twice the radius).

Although the value of π is not exactly equal to any of the numerical expressions which are sometimes used for it, the ratio is very close to 3.14. If extreme accuracy is required, 3.1416 is used as an approximate value of π. Many calculations involving π are satisfactory if the fraction 22/7 is used as the value of π.

Practice problems. Calculate the circumference of each of the following circles, using 22/7 as the value of π:

1. Radius = 21 in.
2. Diameter = 7.28 in.
3. Radius = 14 ft
4. Diameter = 2.8 yd

Answers:

1. 132 in.
2. 22.88 in.
3. 88 ft
4. 8.8 yd

AREA.—The area of a circle is found by multiplying the square of its radius by π. The formula is written as follows:

$$A = \pi r^2$$

EXAMPLE: Find the area of a circle whose diameter is 4 ft, using 3.14 as the value of π.

SOLUTION: The radius is one-half the diameter. Therefore,

$$r = \frac{1}{2}(4 \text{ ft})$$

$$= 2 \text{ ft}$$

$$A = \pi r^2 = \pi(2 \text{ ft})^2$$

$$A = 3.14 \, (4 \text{ sq ft})$$

$$= 12.56 \text{ sq ft}$$

Practice problems. Find the area of each of the following circles, using 3.14 as the value of π.

1. Radius = 7 in.
2. Diameter = 42 mi
3. Diameter = 2.8 ft
4. Radius = 14 yd

Answers:

1. A = 154 sq in.
2. A = 1,385 sq mi
3. 6.15 sq ft
4. 615 sq yd

Concentric Circles

Circles which have a common center are said to be CONCENTRIC. (See fig. 17-20.)

The area of the ring between the concentric circles in figure 17-20 is calculated as follows:

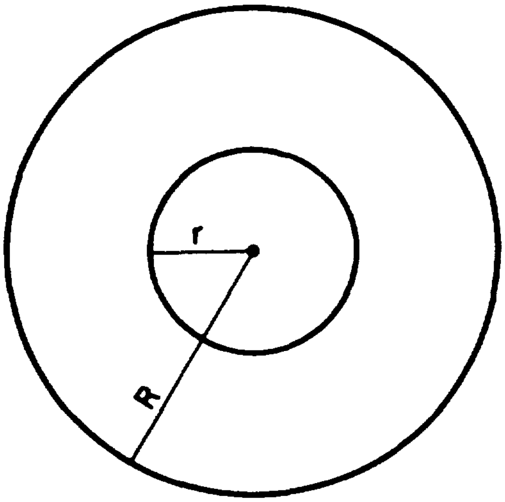

Figure 17-20.—Concentric circles.

Let R = radius of large circle

r = radius of small circle

A_R = area of large circle

A_r = area of small circle

A = area of ring

Then

$$A = A_R - A_r$$

$$= \pi R^2 - \pi r^2$$

$$= \pi(R^2 - r^2)$$

Notice that the last expression is the difference of two squares. Factoring, we have

$$A = \pi(R + r)(R - r)$$

Therefore, the area of a ring between two circles is found by multiplying π times the product of the sum and difference of their radii.

Practice problems. Find the areas of the rings between the following concentric circles:

1. R = 4 in.
   r = 3 in.
2. R = 6 ft
   r = 2 ft

Answers:

1. 22 sq in.
2. 100.5 sq ft